A three-dimensional mathematical transport model for suspended sediment by waves and currents*

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Abstract Based on the equations of momentum and continuity, a general three-dimensional transport model for suspended sediment by waves and currents is developed by respectively decomposing the instantaneous velocities and concentrations into three different time-scale components. The model is composed of two submodels. One is a three-dimensional flow field submodel including the influence of waves, and the other is a three-dimensional diffusion submodel for suspended sediment due to waves and currents. The purpose of this study is to provide a theoretical model for further investigating sediment transport by waves and currents in estuarial and coastal areas.

Keywords: **estuary and coast, three-dimensional model, suspended sediment transport.**

It is a general phenomenon that the interaction occurs among waves, currents and sediments in estuarial and coastal areas. So the sediment transport due to the combined waves and currents has attracted more and more attention. So far, a two-dimensional model for sediment transport by waves and currents has been widely applied[1~8]. In order to describe the three-dimensional character of sediment transport, a quasi-three-dimensional model was used by Cao and Wang[9] to simulate the transport of suspended sediment and the evolution of bed level. Ding et al.[10] derived a three-dimensional diffusion equation which clearly reflects the influence of waves and currents on suspended sediment. And some investigators[11~13] simulated 3-D sediment transport using mixed diffusion coefficients including the influence of waves. But they did not consider the effect of waves on flow field, which is an important factor in flow model, for it is more obvious in shallow-water areas of estuary and coast/

This paper gives the three-dimensional equations of flow field affected by waves, and then derives a more general three-dimensional diffusion equation of suspended sediment by waves and currents. These equations might be very helpful and reasonable because both the waves and currents are taken into account at the same time.

1 Three-dimensional current field equations influenced by waves

Ignoring molecular viscous terms, the three-dimensional basic equations for homogeneous and incompressible fluid may be expressed as

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$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{1}$$

$$\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial uv}{\partial y}+\frac{\partial uw}{\partial z}-fv=-\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{2}$$

$$\frac{\partial v}{\partial t}+\frac{\partial uv}{\partial x}+\frac{\partial v^2}{\partial y}+\frac{\partial vw}{\partial z}+fu=-\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{3}$$

$$\frac{\partial w}{\partial t}+\frac{\partial uw}{\partial x}+\frac{\partial vw}{\partial y}+\frac{\partial w^2}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g, \tag{4}$$

in which f is the Coriolis parameter. To introduce the influence of waves on current field, the velocity field can be decomposed into

$$\begin{aligned} u &= \overline{U}+u_w+u', \\ v &= \overline{V}+v_w+v', \\ w &= \overline{W}+w_w+w', \end{aligned} \tag{5}$$

where $V=(\overline{U}, \overline{V}, \overline{W})$ is a large-scale background velocity field (such as tidal currents, river flow), $\boldsymbol{v}_w=(u_w, v_w, w_w)$ the wave particle velocity field and $\boldsymbol{v}'=(u', v', w')$ the turbulent fluctuation velocity field. Substituting Eq.(5) into Eqs.(1) ~ (4), then implementing time average over a wave period , we can obtain the following equations after a lengthy manipulation:

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}=0, \tag{6}$$

$$\begin{aligned} &\frac{\partial U}{\partial t}+\frac{\partial U^2}{\partial x}+\frac{\partial UV}{\partial y}+\frac{\partial UW}{\partial z}-fV+\frac{\partial}{\partial x}(\overline{u_w^2}-\bar{u}_w^2)+\frac{\partial}{\partial y}(\overline{u_w v_w}-\bar{u}_w\bar{v}_w) \\ &+\frac{\partial}{\partial z}(\overline{u_w w_w}-\bar{u}_w\bar{w}_w)+\frac{\partial}{\partial x}\overline{u'^2}+\frac{\partial}{\partial y}\overline{u'v'}+\frac{\partial}{\partial z}\overline{u'w'}=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x}, \end{aligned} \tag{7}$$

$$\begin{aligned} &\frac{\partial V}{\partial t}+\frac{\partial UV}{\partial x}+\frac{\partial V^2}{\partial y}+\frac{\partial VW}{\partial z}+fU+\frac{\partial}{\partial x}(\overline{u_w v_w}-\bar{u}_w\bar{v}_w)+\frac{\partial}{\partial y}(\overline{v_w^2}-\bar{v}_w^2) \\ &+\frac{\partial}{\partial z}(\overline{v_w w_w}-\bar{v}_w\bar{w}_w)+\frac{\partial}{\partial x}\overline{u'v'}+\frac{\partial}{\partial y}\overline{v'^2}+\frac{\partial}{\partial z}\overline{v'w'}=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial y}, \end{aligned} \tag{8}$$

$$\frac{\partial W}{\partial t}+\frac{\partial UW}{\partial x}+\frac{\partial VW}{\partial y}+\frac{\partial W^2}{\partial z}+\frac{\partial}{\partial z}\frac{\partial}{\partial z}(\overline{w_w^2}-\bar{w}_w^2)+\frac{\partial}{\partial z}\overline{w'^2}=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial z}-g, \tag{9}$$

where $U=\overline{U}+\bar{u}_w$, $V=\overline{V}+\bar{v}_w$, $W=\overline{W}+\bar{w}_w$, denoting the sum of large-scale background currents and wave-induced currents. We assume that the variation of the averaged wave and turbulence situation over a wave period is much smaller in the horizontal direction than in the vertical direction when Eq.(9) is derived. Especially, if wave particle velocity is periodic, meaning that $\bar{u}_w = \bar{v}_w = \bar{w}_w = 0$, then Eqs.(6) ~ (9) may be rewritten as

$$\frac{\partial \overline{U}}{\partial x} + \frac{\partial \overline{V}}{\partial y} + \frac{\partial \overline{W}}{\partial z} = 0, \tag{10}$$

$$\frac{\partial \overline{U}}{\partial t} + \frac{\partial \overline{U}^2}{\partial x} + \frac{\partial \overline{UV}}{\partial y} + \frac{\partial \overline{UW}}{\partial z} - f\overline{V} + \frac{\partial}{\partial x}\overline{u_w^2} + \frac{\partial}{\partial y}\overline{u_w v_w}$$

$$+ \frac{\partial}{\partial z}\overline{u_w w_w} + \frac{\partial}{\partial x}\overline{u'^2} + \frac{\partial}{\partial y}\overline{u'v'} + \frac{\partial}{\partial z}\overline{u'w'} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x}, \tag{11}$$

$$\frac{\partial \overline{V}}{\partial t} + \frac{\partial \overline{UV}}{\partial x} + \frac{\partial \overline{V}^2}{\partial y} + \frac{\partial \overline{VW}}{\partial z} + f\overline{U} + \frac{\partial}{\partial x}\overline{u_w v_w} + \frac{\partial}{\partial y}\overline{v_w^2}$$

$$+ \frac{\partial}{\partial z}\overline{v_w w_w} + \frac{\partial}{\partial x}\overline{u'v'} + \frac{\partial}{\partial y}\overline{v'^2} + \frac{\partial}{\partial z}\overline{v'w'} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial y}, \tag{12}$$

$$\frac{\partial \overline{W}}{\partial t} + \frac{\partial \overline{UW}}{\partial x} + \frac{\partial \overline{VW}}{\partial y} + \frac{\partial \overline{W}^2}{\partial z} + \frac{\partial}{\partial z}\overline{w_w^2} + \frac{\partial}{\partial z}\overline{w'^2} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} - g. \tag{13}$$

In the shallow water area, it may be assumed that $\left|\frac{\mathrm{d}\overline{W}}{\mathrm{d}t}\right| \ll \left|-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial z}\right|$. So Eq. (13) can be simplified into

$$\frac{\partial}{\partial z}\overline{w_w^2} + \frac{\partial}{\partial z}\overline{w'^2} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial z} - g. \tag{14}$$

Integrating Eq. (14) over depth yields

$$\bar{p} = \rho g(\bar{\eta} - z) - \rho\overline{w_w^2} - \rho\overline{w'^2}. \tag{15}$$

Substituting Eq. (15) into Eqs.(11) and (12), we get three-dimensional equations of current field including the wave influence as follows:

$$\frac{\partial \overline{U}}{\partial t} + \frac{\partial \overline{U}^2}{\partial x} + \frac{\partial \overline{UV}}{\partial y} + \frac{\partial \overline{UW}}{\partial z} - f\overline{V}$$

$$= -g\frac{\partial \bar{\eta}}{\partial x} + \left(\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{yx}}{\partial y} + \frac{\partial M_{zx}}{\partial z}\right) + \left(\frac{\partial R_{xx}}{\partial x} + \frac{\partial R_{yx}}{\partial y} + \frac{\partial R_{zx}}{\partial z}\right), \tag{16}$$

$$\frac{\partial \overline{V}}{\partial t} + \frac{\partial \overline{UV}}{\partial x} + \frac{\partial \overline{V}^2}{\partial y} + \frac{\partial \overline{VW}}{\partial z} + f\overline{U} = -g\frac{\partial \bar{\eta}}{\partial y}$$

$$+\left(\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_{yy}}{\partial y}+\frac{\partial M_{zy}}{\partial z}\right)+\left(\frac{\partial R_{xy}}{\partial x}+\frac{\partial R_{yy}}{\partial y}+\frac{\partial R_{zy}}{\partial z}\right), \tag{17}$$

where $M_{xx} = -(\overline{u_w^2} - \overline{w_w^2})$, $M_{xy} = M_{yx} = -\overline{u_w v_w}$, $M_{zx} = -\overline{u_w w_w}$, $M_{zy} = -\overline{v_w w_w}$ and $M_{yy} = -(\overline{v_w^2} - \overline{w_w^2})$ are the momentum flux by waves; $R_{xx} = -(\overline{u'^2} - \overline{w'^2})$, $R_{xy} = R_{yx} = -\overline{u'v'}$, $R_{zx} = -\overline{u'w'}$, $R_{zy} = -\overline{v'w'}$ and $R_{yy} = -(\overline{v'^2} - \overline{w'^2})$ are Reynolds stresses by turbulent fluctuation. In general, parameterizing the Reynolds stresses, the three-dimensional shallow water current field equations influenced by waves can be finally written as

$$\frac{\partial \overline{U}}{\partial t}+\frac{\partial \overline{U}^2}{\partial x}+\frac{\partial \overline{UV}}{\partial y}+\frac{\partial \overline{UW}}{\partial z}-f\overline{V}$$

$$=-g\frac{\partial \overline{\eta}}{\partial x}+\left(\frac{\partial M_{xx}}{\partial x}+\frac{\partial M_{yx}}{\partial y}+\frac{\partial M_{zx}}{\partial z}\right)+\frac{\partial}{\partial x}\left(A_x\frac{\partial \overline{U}}{\partial x}\right)+\frac{\partial}{\partial y}\left(A_y\frac{\partial \overline{U}}{\partial y}\right)+\frac{\partial}{\partial z}\left(A_z\frac{\partial \overline{U}}{\partial z}\right), \tag{18}$$

$$\frac{\partial \overline{V}}{\partial t}+\frac{\partial \overline{UV}}{\partial x}+\frac{\partial \overline{V}^2}{\partial y}+\frac{\partial \overline{VW}}{\partial z}+f\overline{U}$$

$$=-g\frac{\partial \overline{\eta}}{\partial y}+\left(\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_{yy}}{\partial y}+\frac{\partial M_{zy}}{\partial z}\right)+\frac{\partial}{\partial x}\left(A_x\frac{\partial \overline{V}}{\partial x}\right)+\frac{\partial}{\partial y}\left(A_y\frac{\partial \overline{V}}{\partial y}\right)+\frac{\partial}{\partial z}\left(A_z\frac{\partial \overline{V}}{\partial z}\right), \tag{19}$$

where A_x, A_y and A_z are eddy viscosity coefficients.

The boundary conditions of kinematics at sea surface and bottom have the forms of

$$\begin{aligned} \overline{W} &= \frac{\partial \overline{\eta}}{\partial t}+\overline{U}\frac{\partial \overline{\eta}}{\partial x}+\overline{V}\frac{\partial \overline{\eta}}{\partial y}, \quad z=\overline{\eta}(x,y,t), \\ \overline{W} &= -\left(\overline{U}\frac{\partial h}{\partial x}+\overline{V}\frac{\partial h}{\partial y}\right), \quad z=-h(x,y). \end{aligned} \tag{20}$$

Making use of three-dimensional wave solutions and determining eddy viscosity coefficients reasonably, we can get three-dimensional current field solution including wave influence from Eqs. (10) and (18) ~ (20). If the background currents are tidal currents, and the influence of waves on currents is neglected, then $\boldsymbol{v}_w = 0$ in Eq. (5), which means that the momentum flux induced by wave is zero, then Eqs. (18) and (19) become the well-known three-dimensional tidal current equations

$$\frac{\partial \overline{U}}{\partial t}+\frac{\partial \overline{U}^2}{\partial x}+\frac{\partial \overline{UV}}{\partial y}+\frac{\partial \overline{UW}}{\partial z}-f\overline{V}$$

$$=-g\frac{\partial \overline{\eta}}{\partial x}+\frac{\partial}{\partial x}\left(A_x\frac{\partial \overline{U}}{\partial x}\right)+\frac{\partial}{\partial y}\left(A_y\frac{\partial \overline{U}}{\partial y}\right)+\frac{\partial}{\partial z}\left(A_z\frac{\partial \overline{U}}{\partial z}\right), \tag{21}$$

$$\frac{\partial \overline{V}}{\partial t}+\frac{\partial \overline{UV}}{\partial x}+\frac{\partial \overline{V}^2}{\partial y}+\frac{\partial \overline{VW}}{\partial z}+f\overline{U}$$

$$=-g\frac{\partial \bar{\eta}}{\partial y}+\frac{\partial}{\partial x}\left(A_x\frac{\partial \overline{V}}{\partial x}\right)+\frac{\partial}{\partial y}\left(A_y\frac{\partial \overline{V}}{\partial y}\right)+\frac{\partial}{\partial z}\left(A_z\frac{\partial \overline{V}}{\partial z}\right). \tag{22}$$

2 Diffusion equation of suspended sediment by waves and currents

Based on the law of mass conservation, a three-dimensional diffusion equation of suspended sediment can be obtained easily

$$\frac{\partial c}{\partial t}+\frac{\partial uc}{\partial x}+\frac{\partial vc}{\partial y}+\frac{\partial wc}{\partial z}=\frac{\partial \omega_s c}{\partial z}, \tag{23}$$

where c is suspended sediment concentration, and ω_s the settling velocity of sediment particle. Like velocity field, the suspended sediment concentration field can be divided into three components corresponding to the background velocity, waves and turbulence, respectively,

$$c=\overline{C}+c_w+c'. \tag{24}$$

Substituting Eqs. (5) and (24) into Eq. (23), then taking time average over a wave period, we can get a general three-dimensional diffusion equation of suspended sediment due to waves and currents as

$$\frac{\partial C}{\partial t}+\frac{\partial UC}{\partial x}+\frac{\partial VC}{\partial y}+\frac{\partial WC}{\partial z}-\frac{\partial \omega_s C}{\partial z}$$

$$=\frac{\partial \tau_{Rx}}{\partial x}+\frac{\partial \tau_{Ry}}{\partial y}+\frac{\partial \tau_{Rz}}{\partial z}+\frac{\partial \tau_{wx}}{\partial x}+\frac{\partial \tau_{wy}}{\partial y}+\frac{\partial \tau_{wz}}{\partial z}, \tag{25}$$

where $C=\overline{C}+\overline{c_w}$, U, V and W are three-dimensional current field velocities with waves influence taken into account, $\tau_{Rx}=-\overline{u'c'}_s$, $\tau_{Ry}=-\overline{v'c'}_s$, $\tau_{Rz}=-\overline{w'c'}_s$, $\tau_{wx}=-(\overline{u_wc_w}-\overline{u_w}\,\overline{c_w})$, $\tau_{wy}=-(\overline{v_wc_w}-\overline{v_w}\,\overline{c_w})$ and $\tau_{wz}=-(\overline{w_wc_{sw}}-\overline{w_w}\,\overline{c_w})$ denote suspended sediment diffusion by turbulent fluctuation and waves, respectively.

Suppose that wave particle velocity and wave-induced sediment diffusion are periodic, and the parameterization of turbulent diffusion term and wave diffusion term can be represented as

$$\tau_{Rx}=\varepsilon_{cx}\frac{\partial \overline{C}}{\partial x},\ \tau_{Ry}=\varepsilon_{cy}\frac{\partial \overline{C}}{\partial y},\ \tau_{Rz}=\varepsilon_{cz}\frac{\partial \overline{C}}{\partial z}, \tag{26}$$

$$\tau_{wx}=\varepsilon_{wx}\frac{\partial \overline{C}}{\partial x},\ \tau_{wy}=\varepsilon_{wy}\frac{\partial \overline{C}}{\partial y},\ \tau_{wz}=\varepsilon_{wz}\frac{\partial \overline{C}}{\partial z}. \tag{27}$$

Substituting Eqs. (26) and (27) into Eq. (25) yields a three-dimensional diffusion equation of suspended sediment by waves and currents,

$$\frac{\partial \bar{C}}{\partial t}+\frac{\partial \overline{UC}}{\partial x}+\frac{\partial \overline{VC}}{\partial y}+\frac{\partial \overline{WC}}{\partial z}-\frac{\partial \omega_s \bar{C}}{\partial z}$$

$$=\frac{\partial}{\partial x}\left(\varepsilon_x \frac{\partial \bar{C}}{\partial x}\right)+\frac{\partial}{\partial y}\left(\varepsilon_y \frac{\partial \bar{C}}{\partial y}\right)+\frac{\partial}{\partial z}\left(\varepsilon_z \frac{\partial \bar{C}}{\partial z}\right), \tag{28}$$

where $\varepsilon_x = \varepsilon_{cx} + \varepsilon_{wx}$, $\varepsilon_y = \varepsilon_{cy} + \varepsilon_{wy}$ and $\varepsilon_z = \varepsilon_{cz} + \varepsilon_{wz}$ denote the mixed diffusion coefficients by the combined action of turbulent fluctuation and waves, $\overline{U}$, $\overline{V}$ and $\overline{W}$ represent a three-dimensional current field which may be determined by Eqs. (10), (18) ~ (20). Ding et al.[10] gave a similar three-dimensional diffusion equation of suspended sediment by waves and currents, in which the background current field is expressed by $\boldsymbol{V} = (\overline{U}, \overline{V}, 0)$. The present paper develops a more general diffusion equation by taking $\boldsymbol{V} = (\overline{U}, \overline{V}, \overline{W})$. Obviously, if the influence of waves on sediment diffusion is not considered, Eq. (28) can be simplified into a well-known three-dimensional diffusion equation relating only to currents,

$$\frac{\partial \bar{C}}{\partial t}+\frac{\partial \overline{UC}}{\partial x}+\frac{\partial \overline{VC}}{\partial y}+\frac{\partial \overline{WC}}{\partial z}-\frac{\partial \omega_s \bar{C}}{\partial z}$$

$$=\frac{\partial}{\partial x}\left(\varepsilon_{cx} \frac{\partial \bar{C}}{\partial x}\right)+\frac{\partial}{\partial y}\left(\varepsilon_{cy} \frac{\partial \bar{C}}{\partial y}\right)+\frac{\partial}{\partial z}\left(\varepsilon_{cz} \frac{\partial \bar{C}}{\partial z}\right), \tag{29}$$

in which the velocity field may be determined by Equations (10) and (20) ~ (22).

3 Conclusion

The present paper derives a three-dimensional current equation considering the influence of waves and a more general three-dimensional diffusion equation of suspended sediment combining the action of waves and currents in virtue of the law of the conservation of momentum and mass. These equations may be used to investigate three-dimensional characteristics of current field and concentration field, and to study the sediment transport by waves and currents. By the way, the diffusion Eq. (28) in this paper is also suitable for studying other diffusion processes by waves and currents, such as salinity, etc. In addition, it is not difficult to derive a three-dimensional baroclinic current model considering the influence of waves using the method given in this paper. But let it be noted that the present paper only develops a theoretical model; great efforts have to be made to forward practical application. Now the studies on how to calculate the momentum flux by waves, and how to determine the parameters in the model are in progress. Some results will be given in another paper.

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